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RECENT WORK ON AIRFOIL THEORY

By L. Prandtl

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By L. Prandtl

I should like in the following to report briefly on several papers which have appeared in Göttingen during the last three years.

I. In the computational treatment of the lifting surface, progress has been made by starting out - not from a bound vortex distribution on the surface with the associated trailing vortex sheet - but from the acceleration vector field in the neighborhood of the surface (reference 1). Since the acceleration vector $\frac{DW}{dt}$, according to the Euler equation, is equal to $-\frac{1}{\rho}$ grad p and the latter expression for the case of homogeneity of the medium, with compressibility considered, can be written equal to - grad $\int \frac{dp}{\rho}$, the acceleration vector possesses a potential φ, for which there is obtained, by an integration of the Euler equation:

$$\varphi = f(t) - \int \frac{dp}{\rho} \tag{1}$$

For the stationary state, and also for the nonstationary case, if the flow at infinity is free from disturbances, we have in addition, $f(t)$ = constant. Thus there exists a very simple relation between the pressure field and the "acceleration potential." Since the pressure is discontinuous only at the lifting surface and is continuous everywhere else, the same must hold true for the acceleration potential.

II. Where the "linearized" theory is employed (disturbance velocities everywhere small as compared with the flight velocity V), as is customary in the treatment of airfoils, we have:

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*"Ueber neuere Arbeiten zur Theorie der tragenden Fläche." From Proceedings of the Fifth International Congress of Applied Mechanics, Cambridge, Mass., Sept. 1938.

$$\frac{DW}{dt} = \frac{\partial W}{\partial t} + V \frac{\partial W}{\partial x} \tag{2}$$

or, substituting the velocity potential Φ, and the acceleration potential φ:

$$\text{grad } \varphi = \frac{\partial}{\partial t} \text{ grad } \Phi + V \frac{\partial}{\partial x} (\text{grad } \Phi)$$

from which, interchanging differentiation and integration,

$$\varphi = \frac{\partial \Phi}{\partial t} + V \frac{\partial \Phi}{\partial x} \tag{3}$$

In the stationary case $\frac{\partial \Phi}{\partial t} = 0$. It is then possible to obtain Φ from φ by a simple quadrature:

$$\Phi(x, y, z) = \frac{1}{V} \int_{-\infty}^{x} \varphi(x', y, z)\, dx' \tag{4}$$

But also in the general case, Φ can be computed for given φ (x, y, z, t) by an integration of (3). It is necessary to integrate the acceleration potentials impressed on each fluid particle:

$$\Phi\,(x, y, z, t) = \frac{1}{V} \int \varphi \left(x', y, z, t - \frac{x-x'}{V}\right) dx' \tag{4a}$$

The shape of the lifting surface $z = z\,(x, y, t)$ can be obtained from the vertical (downwash) velocity $w = \partial\Phi/\partial z$, from the condition $\frac{\partial z}{\partial x} = \frac{w}{V}$ for the stationary case, or from $\frac{\partial z}{\partial x} + \frac{1}{V} \frac{\partial z}{\partial t} = \frac{w}{V}$ for the nonstationary case by a second quadrature of the same type. For the nonstationary case there is obtained:

$$z = \frac{1}{V} \int_{0}^{x} w\left(x', y, 0, t - \frac{x-x'}{V}\right) dx' + F\left(y, t - \frac{x}{V}\right)$$

The arbitrary function F takes care of an arbitrary lifting above the x,y plane of the wing leading edge, and likewise of an arbitrary vertical motion of the latter.

The vortex sheet does not appear in the above formulation but, of course, exists as is readily seen from (4) or (4a), if consideration is given to the fact that φ is discontinuous at the airfoil and hence in the above integral expressions, these discontinuities also show up behind the airfoil.

III. In our considerations thus far, the Compressibility ~~continuity~~ has not been taken into account. In the linearized theory, it follows from equation (3) that for reasons of continuity the same differential equation that must be satisfied by Φ, must also hold for φ. For the incompressible medium, we thus have, simply:

$$\Delta \varphi = 0 \tag{5}$$

For the compressible medium we have in the stationary case, if the above linearization is again applied,

$$\left(1 - \frac{V^2}{c^2}\right) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{5a}$$

where c is the speed of sound.

With the aid of (5) or (5a), the solution can now be built up by the "doublet distribution" principle known in the electrical theory. A source potential Φ_Q can first be obtained by distributing sources proportional to the intensity of the desired pressure discontinuity at the airfoil or at the neighboring portion of the x,y plane where, for each source at a point $x', y', 0,$ there is to be substituted the singular solution corresponding to differential equation (5) or (5a); thus, in the incompressible case, the solution const/r, where

$$r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}$$

The potential of the source distribution on the unit surface then becomes

$$\Phi_Q = \iint \frac{Q\ (x'\ y')\ dx'\ dy'}{4\ \pi\ r} \tag{6}$$

where Q is the source intensity.

To obtain the velocity potential Φ, it is necessary to pass from the source to the dipole, which can be done by

$$\Phi = \frac{\partial \Phi_Q}{\partial z} \tag{7}$$

IV. Unfortunately, in many applications of the above equations, difficulties arise in carrying out the integrations, so that it is necessary to proceed by the converse method, namely, to find boundaries of the lifting surface, for which computed source potentials are already available. An arbitrary source distribution can then be built up in the form of a series of such source potentials.

In the incompressible case, the procedure has actually been applied for the airfoil with circular plan form. The very complete solution was obtained by W. Kinner (reference 2), who applies elliptic coordinates so that η is constant on confocal ellipsoids of revolution, and μ is constant on the corresponding confocal hyperboloids (see fig. 1), ϑ denoting the azimuth. In these coordinates, the equation $\Delta\varphi = 0$ is satisfied for all

$$\varphi_n^m = P_n^m(\mu)\, Q_n^m(i\eta) \cos m\vartheta, \quad \text{where} \quad P_n^m \quad \text{and} \quad Q_n^m$$

denote, respectively, the spherical harmonics (associated Legendre functions) of the first and second kind. If $n + m$ is odd, φ_n^m has a discontinuity of a type useful for our purpose in the x, y plane within the circle.

With the above functions which tend to zero on the boundary of the circle, and hence give finite velocities there, it is possible to treat cases with impact-free entrance of the flow but not, for example, the case of the flat, circular disk set at an angle to the flow direction - for which case, infinite velocities arise at the leading edge. For this purpose it is necessary to employ additional special functions φ_n, which are obtained from φ_{n+1}^n by a suitable differentiation process. We have

$$\varphi_n = \frac{\mu}{\mu^2 + \eta^2} \left(\frac{1 - \mu^2}{1 + \eta^2}\right)^{n/2} \cos n\vartheta$$

The φ_n functions become infinite over the entire boundary of the circle and therefore do not satisfy the flow condition at the trailing edge where, according to this condition, the disturbance velocity must remain finite. It is possible, however, from the infinitely many functions φ_n^m and φ_n to find such a linear combination that, first, the vertical velocity w is constant in the interior of the circle and, secondly, that the contributions at the trailing edge which become infinite, balance each other, as is possible with the Fourier series. By a combination of the solution for the flat circular disk with the previous solutions, the angle-of-attack variation of any cambered circular surface may also be treated.

The results of this theory have been checked in the wind tunnel for lift, drag, and pressure distribution by M. Hansen, for flat circular plates, spherical segments, and an S-cambered surface with fixed center of pressure. The theoretical drag W_L is, according to the Munk stagger theorem, identical with that of the loaded line. The integral of the pressure distribution over the loaded surface, however, gives a greater drag W_S which, on adding the suction force at the leading edge, reduces to the value of W_L. Since at the edges of thin plates the suction force cannot actually be fully developed, the value of the true drag lies between W_L and W_S, lying nearer the one or the other value, according to the degree of rounding of the edge. The surface with fixed center of pressure (fig. 2) has been so designed by Kinner that, at some angle of attack the flow at the entire leading edge is without impact. At this angle of attack, therefore, W_S and W_L agree.

The entire theory holds only for vanishingly small angles of attack. Deviations of increasing magnitude are therefore to be expected with increasing angle of attack. These appear clearly in the pressure distribution and, naturally, also in the values of the lift and drag. If all the circumstances mentioned are taken into account and consideration is also given to the fact that actually the test surfaces can only be designed with finite thickness, whereas the theory assumes infinitely thin airfoils, the agreement of the results given in figures 3 to 5 may be regarded as satisfactory.

V. An investigation in which the problem for the airfoil with elliptic plan form is to be solved with the aid of ellipsoidal harmonic functions, is now being conducted and likewise, a computation on the nonstationary problem of the vertically flapping circular disk. In the first problem, the lack of certain tables of functions led to difficulties, while in the second the computation appears to proceed smoothly, at least to a first approximation for small frequencies.

The stationary solution for the rectangular airfoil is naturally also of importance. This problem has previously been investigated by H. Blenk (reference 3) by a different method which, however, is applicable only to large aspect-ratio wings (small wing chord), for which in most applications the loaded-line theory is found to be sufficient. It would therefore be of considerable interest to obtain a theory for the rectangular airfoil of small aspect ratio. Unfortunately, however, it appears that the expressions in equations (6) and (7) give rise to insurmountable integration difficulties. The author has therefore directed a computation to be made which is based on the principles of the old airfoil theory and makes use of a large number of loaded lines lying one behind the other. The case of four such loaded lines was first computed, the lift distribution of each of these lines being given by a three-term expression and the kinematic condition $dz/dx = w/V$, being satisfied for discrete points between the lines. A side investigation showed that the accuracy of the computation becomes particularly good if the lines are each located at 1/4 chord of the surface strips, into which the loaded surface is divided, and the points at which the kinematic condition is satisfied are chosen at 3/4 depth of the strips. Our coworker, K. Wieghardt, computed the square plate as a numerical example, and found that the distribution did not deviate much from the elliptic on any of the loaded lines. On the basis of this result and on the assumption that the spanwise lift distribution was accurately elliptic, he also treated the problem for an infinite number of loaded lines, where it was now required that the kinematic condition $w = \text{const}$, be satisfied in the center section only. For the chordwise lift distribution, which is represented by the velocity discontinuity $\gamma(x)$, an integral equation was now obtained, namely,

$$\int_{-1}^{+1} \left\{ \frac{\sqrt{(x'-x)^2 + \lambda^2}}{x' - x} E\left(\frac{\lambda}{\sqrt{(x'-x)^2 + \lambda^2}} \quad \frac{\pi}{2}\right) + \frac{\pi}{2} \right\} \gamma(x)\, dx = 2\pi \lambda V \sin \alpha$$

for all x' between +1 and −1, λ being the aspect ratio b/t, E the elliptic integral. The solution of this integral equation was possible only by numerical methods. For $\gamma(x)$, an expression made up of four "Birnbaum functions" was chosen and the integral equation satisfied at four points. The required quadratures had to be carried out numerically. This computation gave for the various aspect ratios the pressure distributions at the center section shown in figure 6. The agreement of the lifts determined from these pressure distributions with the test results was quite satisfactory, particularly at the small aspect ratios.

VI. With compressibility taken into account and for subsonic speeds, the analogy already given by the author (reference 4) may be directly applied to the preceding results obtained for the incompressible flow. It is necessary to write in formula (6), instead of the previous value of r, the value

$$r = \sqrt{(x - x')^2 + \left(1 - \frac{V^2}{c^2}\right)\left((y - y')^2 + z^2\right)}$$

hence the space filled with the compressible flow can be set to correspond with an affine space filled with a corresponding incompressible flow by reducing all dimensions in the direction of the x axis; for example, in the ratio $1 - V^2/c^2$, while the dimensions in the y and z directions remain unchanged (Prandtl-Glauert rule).

The case of supersonic velocity was treated in detail by H. Schlichting (reference 5) by the new methods. It was here possible to solve completely the case of the flat rectangular plate - with the restriction, however, that the disturbance regions which spread out at both side edges under the Mach angle do not overlap on the surface. The considerations which have led to formula (6) can here be applied again by setting in the formula

$$r = \sqrt{(x - x')^2 - \left(\frac{V^2}{c^2} - 1\right)(y - y')^2 + z^2}$$

The values are real only within a double cone with the vertex at x', y', 0, while outside they are imaginary. The physical sense requires, however, that $Q(x', y')$ should contribute only in the after cone, while in the forward cone and in the outside region $1/r$ must be set identically equal to zero. In order that the forward cone should drop out, it is necessary to add a factor 2 to the after cone.

The supersonic problems are simpler than the incompressible flow and subsonic problems, in that at the leading edges no flow arises with infinite velocity (and hence no suction force); also, at the trailing edges no pressure difference approaching zero is required. Practical partial solutions even with the simplest integral functions are therefore obtained for the distribution of the lift density. Thus Schlichting, for example, has computed the velocity fields for the uniformly loaded rectangular, triangular, and trapezoidal wings. The trapezoidal wing constitutes an important preliminary work for the theory of the flat rectangular plate set at angle α to the flow direction. In this case the following situation is obtained (fig. 7). In a trapezoid ABFE, whose sides AE and BF are inclined at the Mach angle β, the lift density is constant, being equal approximately to $4\,\alpha \tan\beta\, \rho V^2/2$ (where the Mach angle α is given by $\sin\alpha = c/V$). Toward the side edges the drop takes place in such a manner that along each straight line inclined at an angle φ, for example, BG, the lift density is constant. The entire lift distribution can therefore be built up by a superposition of a large number of uniformly loaded trapezoidal wings with various angles φ. In the limit this leads to an integral equation which - by a substitution found by I. Lotz - can finally be brought to the following form: Let $\vartheta = \tan\varphi/\tan\alpha$, and $f(\vartheta)$ be the lift density measured at the angle φ in the trapezoid-shaped center field. Then

$$F(\vartheta) + \frac{1}{\pi}\int_{\theta=0}^{\theta=1} F'(\theta)\, F(\vartheta,\theta)\, d\theta = 1$$

with

$$F(\vartheta,\theta) = \int_{\vartheta'=1}^{\vartheta'=\vartheta} \frac{\sqrt{1-\vartheta'^2}\, d\vartheta'}{\vartheta'(\vartheta'-\theta)}$$

The integral equation is to be solved for the boundary conditions $f(\vartheta) = 0$ for $\vartheta = 0$ and $f(\vartheta) = 1$ for $\vartheta = 1$, The solution has been carried out numerically by Miss I. Lotz and J. Pretsch in reference 5. Figure 7 gives the results of the computation.

SUMMARY

The basic ideas of a new method for treating the problem of the airfoil are presented, and a review is given of the problems thus far computed for incompressible and supersonic flows. Test results are reported for the airfoil of circular plan form and the results are shown to agree well with the theory. As a supplement, a theory based on the older methods is presented for the rectangular wing of small aspect ratio.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

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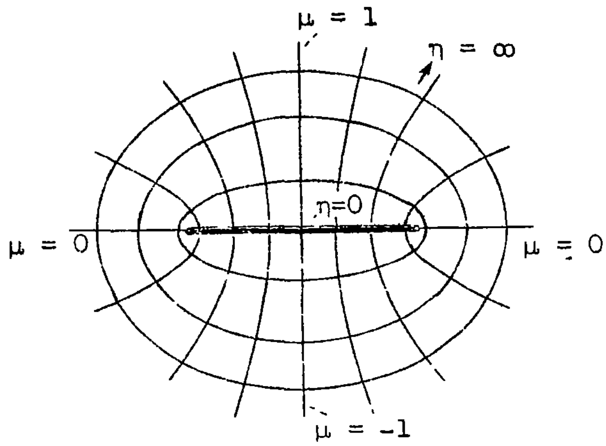

Figure 1.- Elliptic coordinates

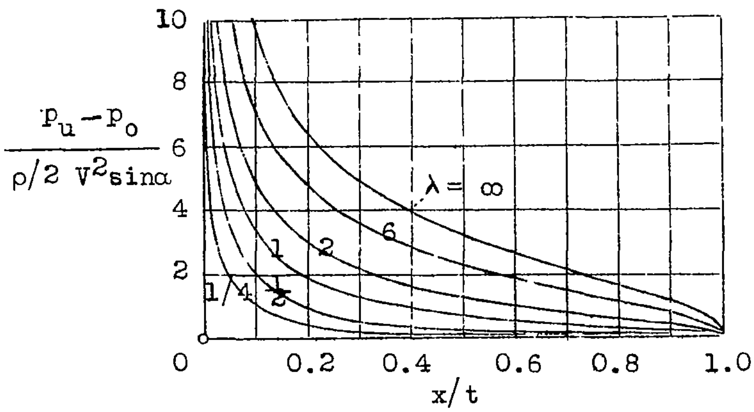

Figure 6.- Lift distribution in the center section of large chord flat rectangular plates, after K. Wieghardt (λ = aspect ratio).

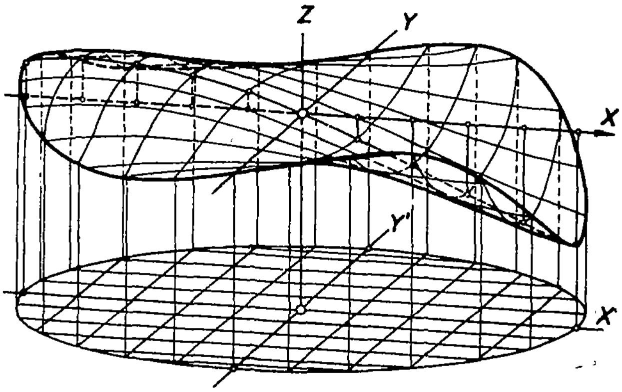

Figure 2.- Surface of circular plan form with fixed center of pressure, after W. Kinner (flow from the left).

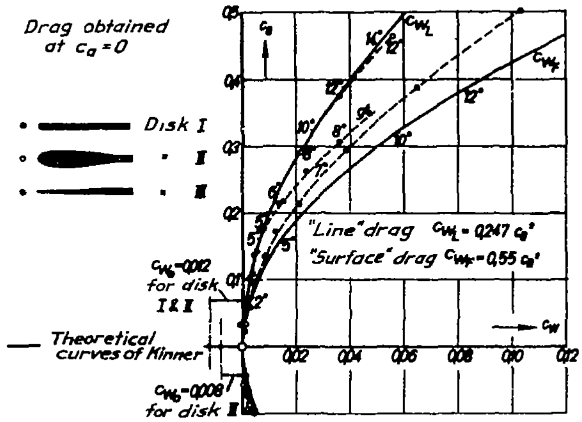

Figure 5.- Polar curves of flat circular disks with rounded leading edge.

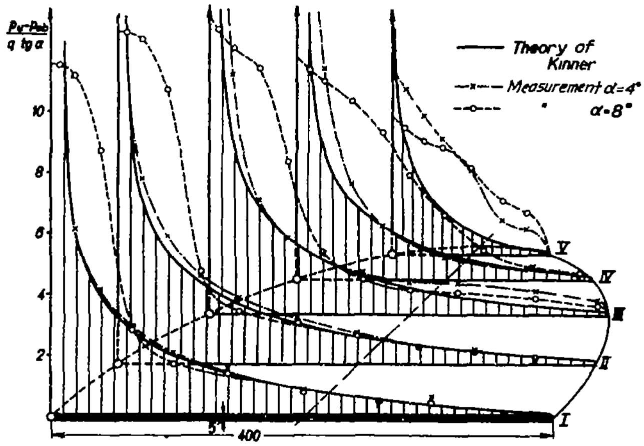

Figure 3.- Pressure distribution over flat circular disk set at an angle(the measured values are divided by the dynamic pressure q and tan α).

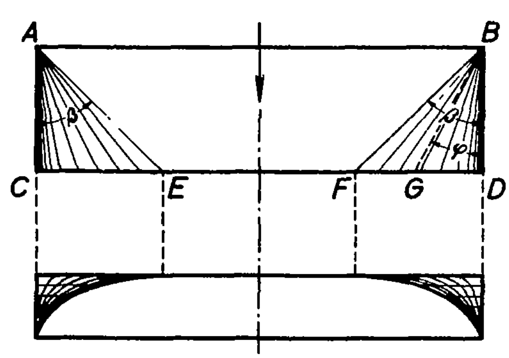

Figure 7.- Pressure distribution on the flat rectangular plate at supersonic speeds, after H. Schlichting and I.Lotz.

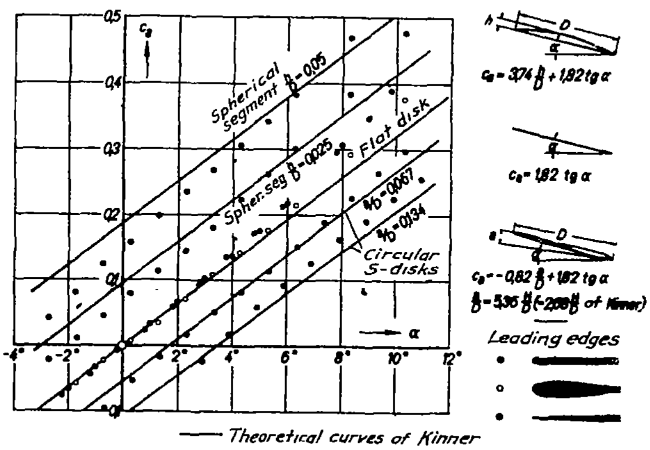

Figure 4.- Lift coefficients of circular airfoils of various camber and rounding of the leading edge as a function of the angle of attack.